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Minimal infinite submodule-closed subcategories

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Abstract

Let A be an artin algebra. We are going to consider full subcategories of $\text{mod } A$ closed under finite direct sums and under submodules with infinitely many isomorphism classes of indecomposable modules. The main result asserts that such a subcategory contains a minimal one and we exhibit some striking properties of these minimal subcategories. These results have to be considered as essential finiteness conditions for such module categories.

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1. Introduction

Let A be an artin algebra, and $\text{mod } A$ the category of A -modules of finite length. All the subcategories to be considered will be full subcategories of $\text{mod } A$ closed under isomorphisms, finite direct sums and direct summands, but note that we also consider individual A -modules which may not be of finite length. If the A module X has finite length, we denote its length by $|X|$.

Let \mathcal{C} be a subcategory of $\text{mod } A$. We say that \mathcal{C} is *finite* provided it contains only finitely many isomorphism classes of indecomposable modules, otherwise \mathcal{C} is said to be *infinite*. Of course, \mathcal{C} is said to be *submodule-closed* provided for any module C in \mathcal{C} also any submodule of C belongs to \mathcal{C} .

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The aim of this paper is to study infinite submodule-closed subcategories of $\text{mod } \Lambda$. A subcategory \mathcal{C} of $\text{mod } \Lambda$ will be called *minimal infinite submodule-closed*, or (in this paper) just *minimal*, provided it is infinite and submodule-closed, and no proper subcategory of \mathcal{C} is both infinite and submodule-closed. On a first thought, it is not at all clear whether minimal subcategories do exist: the existence is in sharp contrast to the usual properties of infinite structures (recall that in set theory, a set is infinite iff it contains proper subsets of the same cardinality).

Theorem 1. *Any infinite submodule-closed subcategory of $\text{mod } \Lambda$ contains a minimal subcategory.*

Of course, the assertion is of interest only in case Λ is representation-infinite. But already the special case of looking at the category $\text{mod } \Lambda$ itself, with Λ representation-infinite, should be stressed: *The module category of any representation-infinite artin algebra has minimal subcategories.*

Let M be a Λ -module, not necessarily of finite length. We write \mathcal{S}_M for the class of finite length modules cogenerated by M . This is clearly a submodule-closed subcategory of $\text{mod } \Lambda$. (Conversely, any submodule-closed subcategory \mathcal{C} of $\text{mod } \Lambda$ is of this form: take for M the direct sum of all modules in \mathcal{C} , one from each isomorphism class; or else, it is sufficient to take just indecomposable modules in \mathcal{C} .)

Theorem 2. *Let \mathcal{C} be a minimal subcategory of $\text{mod } \Lambda$. Then*

- (a) *For any natural number d , there are only finitely many isomorphism classes of indecomposable modules in \mathcal{C} of length at most d and even only finitely many isomorphism classes of indecomposable modules which are submodules of a direct sum of modules C_i in \mathcal{C} with $|C_i| \leq d$.*
- (b) *Any module in \mathcal{C} is isomorphic to a submodule of an indecomposable module in \mathcal{C} .*
- (c) *There is an infinite sequence of indecomposable modules C_i in \mathcal{C} with proper inclusions*

$$C_1 \subset C_2 \subset \cdots \subset C_i \subset C_{i+1} \subset \cdots$$

such that also the union $M = \bigcup_i C_i$ is indecomposable and then $\mathcal{C} = \mathcal{S}_M$.

As we have mentioned, Theorem 1 asserts, in particular, that the module category of any representation-infinite artin algebra has a minimal subcategory \mathcal{C} , and the assertion (c) of Theorem 2 yields arbitrarily large indecomposable modules in \mathcal{C} . This shows that we are in the realm of the first Brauer–Thrall conjecture (formulated by Brauer and Thrall around 1940 and proved by Roiter in 1968): any representation-infinite artin algebra has indecomposable modules of arbitrarily large length. The proof of Roiter and its combinatorial interpretation by Gabriel are the basis of the Gabriel–Roiter measure on $\text{mod } \Lambda$, see [3,4,8]. Using it, we have shown in [3] that the module category of a representation-infinite artin algebra always has a so-called take-off part: this is an infinite submodule-closed subcategory with property (a) of Theorem 2, and there is an infinite inclusion chain of indecomposables such that also the union M is indecomposable, as in property (c) of Theorem 2. However, \mathcal{S}_M usually will be a proper subcategory of the take-off part, and then the take-off part cannot be minimal. Of course, we can apply Theorem 1 to the take-off part in order to obtain a minimal subcategory inside the take-off part. The important feature of the minimal categories is the following: we deal with a countable set of indecomposable

modules which are strongly interlaced as the assertions (b) and (c) of Theorem 2 assert. Typical examples to have in mind are the infinite preprojective components of hereditary algebras (see Section 4).

The proof of Theorem 1 will be given in Section 2, the proof of Theorem 2 in Section 3. These proofs depend on the Gabriel–Roiter measure for Λ -modules, as discussed in [3,4]. The remaining Section 4 provides examples. First, we will mention some procedures for obtaining submodule-closed subcategories. Then, following Kerner–Takane, we will show that the preprojective component of a representation-infinite connected hereditary algebra Λ is always a minimal subcategory. In case Λ is tame, this is the only one, but for wild hereditary algebras, there will be further ones.

2. Proof of Theorem 1

Given a class \mathcal{X} of modules of finite length (or of isomorphism classes of modules), we denote by $\text{add } \mathcal{X}$ the smallest subcategory containing \mathcal{X} . We denote by $\mathbb{N} = \mathbb{N}_1$ the natural numbers starting with 1.

The proof will be based on results concerning the Gabriel–Roiter measure for Λ -modules, see [3,4]. For the benefit of the reader, let us recall the inductive definition of the *Gabriel–Roiter measure* $\mu(M)$ of a Λ -module M : For the zero module $M = 0$, one sets $\mu(0) = 0$. If $M \neq 0$ is decomposable, then $\mu(M)$ is the maximum of $\mu(M')$ where M' is a proper submodule of M , whereas for an indecomposable module M , one sets

$$\mu(M) = 2^{-|M|} + \max_{M' \subset M} \mu(M').$$

If M is indecomposable and not simple, then there always exists an indecomposable submodule $M' \subset M$ such that $\mu(M) - \mu(M') = 2^{-|M|}$, such submodules are called *Gabriel–Roiter submodules* of M . Inductively, we obtain for any indecomposable module M a chain of indecomposable submodules

$$M_1 \subset M_2 \subset \cdots \subset M_{t-1} \subset M_t = M$$

such that M_1 is simple and M_{i-1} is a Gabriel–Roiter submodule of M_i , for $2 \leq i \leq t$. Note that

$$\mu(M) = \sum_{j=1}^t 2^{-|M_j|},$$

and it will sometimes be convenient to call also the set $I = \{|M_1|, \dots, |M_t|\}$ the Gabriel–Roiter measure of M . Thus the Gabriel–Roiter measure $\mu(M)$ of a module M will be considered either as a finite set I of natural numbers, or else as the rational number $\sum_{i \in I} 2^{-i}$, whatever is more suitable.

Given a subcategory \mathcal{C} of $\text{mod } \Lambda$ and a finite set I of natural numbers, let $\mathcal{C}(I)$ be the set of isomorphism classes of indecomposable objects in \mathcal{C} with Gabriel–Roiter measure I . An obvious adaption of one of the main results of [3] asserts:

There is an infinite sequence of Gabriel–Roiter measures $I_1 < I_2 < \cdots$ such that $\mathcal{C}(I_t)$ is non-empty for any $t \in \mathbb{N}$ and such that for any J with $\mathcal{C}(J) \neq \emptyset$, either $J = I_t$ for some t or else $J > I_t$ for all t . Moreover, all the sets $\mathcal{C}(I_t)$ are finite. (Note that the sequence of measures I_t depends on \mathcal{C} , thus one should write $I_t^{\mathcal{C}} = I_t$; the papers [3,4] were dealing only with the case $\mathcal{C} = \text{mod } \Lambda$, but the proofs carry over to the more general case of dealing with a submodule-closed subcategory \mathcal{C}).

Since $\text{add} \bigcup_{t \in \mathbb{N}} \mathcal{C}(I_t)$ is an infinite submodule-closed subcategory of \mathcal{C} , we may assume that $\mathcal{C} = \text{add} \bigcup_{t \in \mathbb{N}} \mathcal{C}(I_t)$. In order to construct a minimal subcategory \mathcal{C}' , we will construct a sequence of subcategories

$$\mathcal{C} = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \cdots$$

with the following properties:

- (a) Any subcategory \mathcal{C}_i is infinite and submodule-closed,
- (b) $\mathcal{C}_i(I_t) = \mathcal{C}_t(I_t)$ for $t \leq i$.
- (c) If $\mathcal{D} \subseteq \mathcal{C}_i$ is infinite and submodule-closed, then

$$\mathcal{D}(I_t) = \mathcal{C}_t(I_t) \quad \text{for } t \leq i.$$

We start with $\mathcal{C}_0 = \mathcal{C}$ (the t in conditions (b) and (c) satisfies $t \geq 1$, thus nothing has to be verified). Assume, we have constructed \mathcal{C}_i for some $i \geq 0$, satisfying the conditions (a), and the conditions (b), (c) for all pairs (i, t) with $t \leq i$. We are going to construct \mathcal{C}_{i+1} .

Call a subset \mathcal{X} of $\mathcal{C}_i(I_{i+1})$ *good*, provided there is a subcategory $\mathcal{D}_{\mathcal{X}}$ of \mathcal{C}_i which is infinite and submodule-closed and such that $\mathcal{D}_{\mathcal{X}}(I_{i+1}) = \mathcal{X}$. For example $\mathcal{C}_i(I_{i+1})$ itself is good (with $\mathcal{D}_{\mathcal{X}} = \mathcal{C}_i$). Since $\mathcal{C}_i(I_{i+1})$ is a finite set, we can choose a minimal good subset $\mathcal{X}' \subseteq \mathcal{C}_i(I_{i+1})$. For \mathcal{X}' , there is an infinite and submodule-closed subcategory $\mathcal{D}_{\mathcal{X}'}$ of \mathcal{C}_i such that $\mathcal{D}_{\mathcal{X}'}(I_{i+1}) = \mathcal{X}'$. (Note that in general neither \mathcal{X}' nor $\mathcal{D}_{\mathcal{X}'}$ will be uniquely determined: usually, there may be several possible choices.) Let $\mathcal{C}_{i+1} = \mathcal{D}_{\mathcal{X}'}$. By assumption, \mathcal{C}_{i+1} is infinite and submodule-closed, thus (a) is satisfied. In order to show (b) for all pairs $(i+1, t)$ with $t \leq i+1$, we first consider some $t \leq i$. We can apply (c) for $\mathcal{D} = \mathcal{C}_{i+1} \subseteq \mathcal{C}_i$ and see that $\mathcal{D}(I_t) = \mathcal{C}_t(I_t)$, as required. But for $t = i+1$, nothing has to be shown. Finally, let us show (c). Thus let $\mathcal{D} \subseteq \mathcal{C}_{i+1}$ be an infinite submodule-closed subcategory. Since $\mathcal{D} \subseteq \mathcal{C}_i$, we know by induction that $\mathcal{D}(I_t) = \mathcal{C}_t(I_t)$ for $t \leq i$. It remains to show that $\mathcal{D}(I_{i+1}) = \mathcal{C}_{i+1}(I_{i+1})$. Since $\mathcal{D} \subseteq \mathcal{C}_{i+1}$, we have $\mathcal{D}(I_{i+1}) \subseteq \mathcal{C}_{i+1}(I_{i+1})$. But if this would be a proper inclusion, then $\mathcal{X} = \mathcal{D}(I_{i+1})$ would be a good subset of $\mathcal{C}_i(I_{i+1})$ which is properly contained in $\mathcal{C}_{i+1}(I_{i+1}) = \mathcal{D}_{\mathcal{X}'}(I_{i+1})$, a contradiction to the minimality of \mathcal{X}' . This completes the inductive construction of the various \mathcal{C}_i .

Now let

$$\mathcal{C}' = \bigcap_{i \in \mathbb{N}} \mathcal{C}_i.$$

Of course, \mathcal{C}' is submodule-closed. Also, we see immediately

$$(b') \quad \mathcal{C}'(I_t) = \mathcal{C}_t(I_t) \quad \text{for all } t,$$

since $\mathcal{C}'(I_t) = \bigcap_{i \geq t} \mathcal{C}_i(I_t) = \mathcal{C}_t(I_t)$, according to (b).

First, we show that \mathcal{C}' is infinite. Of course, $\mathcal{C}'(I_1) \neq \emptyset$, since $I_1 = \{1\}$ and a good subset of $\mathcal{C}_0(I_1)$ has to contain at least one simple module. Assume that $\mathcal{C}'(I_s) \neq \emptyset$ for some s , we want to see that there is $t > s$ with $\mathcal{C}'(I_t) \neq \emptyset$. For every Gabriel–Roiter measure I , let $n(I)$ be the minimal number n with $I \subseteq [1, n]$, thus $n(I)$ is the length of the modules in $\mathcal{C}(I)$. Let $n(s)$ be the maximum of $n(I_j)$ with $j \leq s$, thus $n(s)$ is the maximal length of the modules in $\bigcup_{j \leq s} \mathcal{C}(I_j)$. Let s' be a natural number such that $n(I_j) > n(s)pq$ for all $j > s'$ (such a number

exists, since the modules in $\mathcal{C}(I_j)$ with j large, have large length); here p is the maximal length of an indecomposable projective module, q that of an indecomposable injective module.

We claim that $\mathcal{C}'(I_j) \neq \emptyset$ for some j with $s < j \leq s'$. Assume for the contrary that $\mathcal{C}'(I_j) = \emptyset$ for all $s < j \leq s'$. We consider $\mathcal{C}_{s'}$. Since $\mathcal{C}_{s'}$ is infinite, there is some $t > s$ with $\mathcal{C}_{s'}(I_t) \neq \emptyset$, and we choose t minimal. Now for $s < j \leq s'$, we know that $\mathcal{C}_{s'}(I_j) = \mathcal{C}_j(I_j) = \mathcal{C}'(I_j) = \emptyset$, according to (b) and (b'). This shows that $t > s'$. Let Y be an indecomposable module with isomorphism class in $\mathcal{C}_{s'}(I_t)$. Let X be a Gabriel–Roiter submodule of Y . Then X belongs to $\mathcal{C}_{s'}(I_j)$ with $j < t$. If $j \leq s$, then the length of X is bounded by $n(s)$, and therefore Y is bounded by $n(s)pq$ (see [4], 3.1 Corollary), in contrast to the fact that $n(I_t) > n(s)pq$. This is the required contradiction. Thus \mathcal{C}' is infinite.

Now, let \mathcal{D} be an infinite submodule-closed subcategory of \mathcal{C}' . We show that $\mathcal{D}(I_t) = \mathcal{C}'(I_t)$ for all t . Consider some fixed t and choose an i with $i \geq t$. Since $\mathcal{C}' \subseteq \mathcal{C}_i$, we see that $\mathcal{D}(I_t) = \mathcal{C}_i(I_t)$ for the given t , according to (b) for \mathcal{C}_i . But according to (b'), we also know that $\mathcal{C}'(I_t) = \mathcal{C}_i(I_t)$. This completes the proof.

3. Proof of Theorem 2

We refer to [3] for the proof of (a) and for the construction of an inclusion chain

$$C_1 \subset C_2 \subset \cdots \subset C_i \subset C_{i+1} \subset \cdots$$

with indecomposable union, as asserted in (c). In [3] these assertions have been shown for the take-off part of $\text{mod } \Lambda$, but the same proof with only minor modifications, carries over to minimal categories.

To complete the proof of (c), we only have to note the following: By construction, \mathcal{S}_M contains all the modules C_i , thus \mathcal{S}_M is not finite. But of course, $\mathcal{S}_M \subseteq \mathcal{C}$. Namely, if X is a finite length module which is cogenerated by M , then there are finitely many maps $f_i : X \rightarrow M$ such that the intersection of the kernels is zero. But there is some j such that the images of all the maps f_i are contained in C_j , therefore X is cogenerated by C_j and thus belongs to \mathcal{C} . The minimality of \mathcal{C} implies that $\mathcal{S}_M = \mathcal{C}$.

It remains to proof part (b) of Theorem 2. We will need some general observations which may be of independent interest. Recall that a module is said to be of *finite type*, provided it is the direct sum of (may-be infinitely many) copies of a finite number of modules of finite length.

(1) *If \mathcal{S}_M is minimal, then M is not of finite type.*

Proof. Assume that M is of finite type, let M_1, \dots, M_t be the indecomposable direct summands of M , one from each isomorphism class. We may assume that they are indexed with increasing Gabriel–Roiter measures, thus $\mu(M_i) \leq \mu(M_j)$ for $i \leq j$. Let \mathcal{S}' be the subcategory of $\text{mod } \Lambda$ such that N belongs to \mathcal{S}' if and only if any indecomposable direct summand of N belongs to \mathcal{S}_M and is not isomorphic to M_t . Thus \mathcal{S}' is a proper subcategory of \mathcal{S}_M and infinite. We claim that \mathcal{S}' is submodule closed (this then contradicts the minimality of \mathcal{S}_M).

Let N be in \mathcal{S}' . We want to show that any indecomposable submodule U of N belongs to \mathcal{S}' . Since $\mathcal{S}' \subset \mathcal{S}_M$, we know that U belongs to \mathcal{S}_M , thus we have to exclude that U is isomorphic to M_t . Thus, let us assume that $U = M_t$ and let $u : M_t \rightarrow N$, be the embedding. Since N belongs to \mathcal{S}_M , there is an embedding $u' : N \rightarrow M^r$ for some r . Altogether, there is the embedding $u'u : M_t \rightarrow M^r$. Now $\mu(M_t) = \max_{1 \leq i \leq r} \mu(M_i)$, and therefore $u'u$ is a split monomorphism.

Consequently, also $u : M_t \rightarrow N$ is a split monomorphism. But since $N \in \mathcal{S}'$, no direct summand of N is isomorphic to M_t . This shows that \mathcal{S}' is submodule closed. \square

(2) If \mathcal{S}_M is minimal and $M' \subseteq M$ is a cofinite submodule, then $\mathcal{S}_{M'} = \mathcal{S}_M$.

Proof. Of course, $\mathcal{S}_{M'} \subseteq \mathcal{S}_M$. Since we assume that \mathcal{S}_M is minimal, we only have to show that $\mathcal{S}_{M'}$ is infinite. Assume, for the contrary, that $\mathcal{S}_{M'}$ is finite. This implies that M' is of finite type (see [7]), say $M' = \bigoplus_{i \in I} M'_i$, so that the modules M'_i belong to only finitely many isomorphism classes. Let U be a submodule of M of finite length such that $M' + U = M$. Now $M' \cap U$ is a submodule of M' of finite length, thus it is contained in some $M'' = \bigoplus_{i \in J} M'_i$, where J is a finite subset of I . Of course, $M'' + U$ is a submodule of finite length. We claim that $M = (M'' + U) \oplus M'''$, where $M''' = \bigoplus_{i \in I \setminus J} M'_i$. Namely, on the one hand, $M'' + U + M''' = M' + U = M$, whereas, on the other hand, $(M'' + U) \cap M''' \subseteq (M'' + U) \cap M' = M'' + (U \cap M') \subseteq M''$, thus $(M'' + U) \cap M'''$ is contained both in M'' and M''' , therefore in $M'' \cap M''' = 0$. Since both modules $M'' + U$ and M''' are of finite type, also $M = (M'' + U) \oplus M'''$ is a module of finite type. But this contradicts (1). \square

(3) Assume that $\mathcal{C} = \mathcal{S}_M$ is minimal and let M_0 be a submodule of M of finite length. If X belongs to \mathcal{C} , then there is an embedding $u : X \rightarrow M$ such that $M_0 \cap u(X) = 0$.

Proof. Let X be of finite length and cogenerated by M . We want to construct inductively maps $f : X \rightarrow M$ such that $M_0 \cap f(X) = 0$ and such that the length of $\text{Ker}(f)$ decreases. As start, we take as f the zero map. The process will end when $\text{Ker}(f) = 0$.

Thus, assume that we have given some map $f : X \rightarrow M$ with $M_0 \cap f(X) = 0$ and $\text{Ker}(f) \neq 0$. We are going to construct a map $g : X \rightarrow M$ such that first $M_0 \cap g(X) = 0$ and second, $\text{Ker}(g)$ is a proper submodule of $\text{Ker}(f)$. Let $M_1 = M_0 + f(X)$, this is a submodule of finite length of M . Choose a submodule M' of M with $M_1 \cap M' = 0$, and maximal with this property. Note that M' is a cofinite submodule of M (namely, M/M' embeds into the injective hull of M_1 , and with M_1 also its injective hull has finite length). According to (2), we know that $\mathcal{S}_{M'} = \mathcal{S}_M = \mathcal{C}$, thus X belongs to $\mathcal{S}_{M'}$. This means that X is cogenerated by M' . In particular, since $\text{Ker}(f) \neq 0$, there is a map $f' : X \rightarrow M'$ such that $\text{Ker}(f)$ is not contained in $\text{Ker}(f')$. Let $g = (f, f') : X \rightarrow M_1 \oplus M' \subseteq M$. Then $\text{Ker}(g) = \text{Ker}(f) \cap \text{Ker}(f')$ is a proper submodule of $\text{Ker}(f)$. Also, the image $g(X)$ is contained in $f(X) + f'(X) \subseteq f(X) + M'$. Since $M_1 + M' = M_0 \oplus f(X) \oplus M'$, we see that $M_0 \cap g(X) = 0$.

This completes the induction step. After finitely many steps, we obtain in this way an embedding u of X into M such that $u(X) \cap M_0 = 0$. \square

(3') Assume that $\mathcal{C} = \mathcal{S}_M$ is minimal. If X, Y are submodules of M of finite length, then also $X \oplus Y$ is isomorphic to a submodule of M .

Proof. If X, Y are submodules of M , then $X \oplus Y$ is cogenerated by M . \square

(3'') Assume that $\mathcal{C} = \mathcal{S}_M$ is minimal. If C belongs to \mathcal{C} , then the direct sum of countably many copies of C can be embedded into M .

Proof. Assume, there is given an embedding $u_t : C^t \rightarrow M$, where $t \geq 0$ is a natural number. Let $M_0 = u_t(C^t)$. According to (3), we find an embedding $u : C \rightarrow M$ such that $M_0 \cap u(C) = 0$. Thus, let $u_{t+1} = u_t \oplus u : C^{t+1} = C^t \oplus C \rightarrow M$. \square

Proof of part (b) of Theorem 2. Let C be a module in \mathcal{C} . Let $M = \bigcup_i C_i$ be as constructed in (c), thus all the C_i are indecomposable and $\mathcal{S}_M = \mathcal{C}$. According to (3), there is an embedding $u : C \rightarrow M$. Now the image of u lies in some C_i , thus u embeds C into the indecomposable module C_i . \square

Some consequences of Theorem 2(b) should be mentioned. If S is a simple Λ -module, write $[X : S]$ for the Jordan–Hölder multiplicity of S in the Λ -module X .

Corollary 1. *Let \mathcal{C} be a minimal subcategory. For any natural number d , there is an indecomposable module C in \mathcal{C} with the following property: if S is a simple Λ -module with $[Y : S] \neq 0$ for some Y in \mathcal{C} , then $[C : S] \geq d$.*

Proof. We consider the simple Λ -modules S such that there exists a module $Y(S)$ in \mathcal{C} with $[Y(S) : S] \neq 0$, and let $Y = \bigoplus Y(S)$ where the summation extends over all isomorphism classes of such simple modules S . Given a natural number d , let us consider Y^d . According to assertion (b) of Theorem 2, there is an indecomposable Λ -module C such that Y^d embeds into C . But this implies that $[C : S] \geq [Y^d : S] = d[Y : S] \geq d[Y(S) : S] \geq d$. \square

Note that the corollary provides a strengthening of the assertion of the first Brauer–Thrall conjecture:

Corollary 2. *Let Λ be representation-infinite. Let $P = Ae$ be indecomposable projective (e an idempotent in Λ) and $S = P/\text{rad } P$. If $[M : S]$ is bounded for the indecomposable modules M , then $\Lambda/\langle e \rangle$ is representation-infinite.*

Proof. Take a minimal subcategory \mathcal{C} of $\text{mod } \Lambda$ and let \mathcal{I} be its annihilator. Let $\Lambda' = \Lambda/\mathcal{I}$, thus \mathcal{C} is a minimal subcategory of $\text{mod } \Lambda'$ and for every simple Λ' -module S , there is a Λ' -module Y with $[Y : S] \neq 0$. By Corollary 1, the numbers $[C : S]$ with C indecomposable in \mathcal{C} is unbounded. If $e \notin \mathcal{I}$, then S is a simple Λ' -module and then $[C : S]$ with C indecomposable in \mathcal{C} is unbounded. But this contradicts the assumption on S . Thus we see that $e \in \mathcal{I}$, therefore Λ' is a factor algebra of $\Lambda/\langle e \rangle$. Since Λ' is representation-infinite, also $\Lambda/\langle e \rangle$ is representation-infinite. \square

Corollary 3. *A representation-infinite artin algebra has indecomposable representations X such that all non-zero Jordan–Hölder multiplicities of X are arbitrarily large.*

4. Examples

First, let us mention some ways for obtaining submodule-closed subcategories.

- Of course, we can consider the module category $\text{mod } \Lambda$ itself.
- If \mathcal{I} is a two-sided ideal of Λ , then the Λ -modules annihilated by \mathcal{I} form a submodule-closed subcategory (this subcategory is just the category of all Λ/\mathcal{I} -modules).

- As we have mentioned in Section 3, we may start with an arbitrary (not necessarily finitely generated) module M , and consider the subcategory \mathcal{S}_M of all finite length modules co-generated by M . This subcategory \mathcal{S}_M is submodule-closed, and any submodule-closed subcategory of $\text{mod } \Lambda$ is obtained in this way.
- The special case of dealing with $M = {}_\Lambda \Lambda$ has been studied often in representation theory; the modules in \mathcal{S}_Λ are called the *torsionless* Λ -modules. Artin algebras with \mathcal{S}_Λ finite have quite specific properties, for example their representation dimension is bounded by 3.
- The categories $\mathcal{A}(<\gamma)$ and $\mathcal{A}(\leq \gamma)$ of all modules X in $\mathcal{A} = \text{mod } \Lambda$ with Gabriel–Roiter measure $\mu(X) < \gamma$, or $\mu(X) \leq \gamma$, respectively; here $\gamma \in \mathbb{R}$ and μ is the Gabriel–Roiter measure (or a weighted Gabriel–Roiter measure).
- In particular, the take-off subcategory of $\text{mod } \Lambda$ (as introduced in [3]) is submodule-closed (and it is infinite iff Λ is representation-infinite).
- If Λ has global dimension n , then the subcategory \mathcal{C} of all modules of projective dimension at most $n - 1$ is closed under cogeneration (and extensions) (this is mentioned for example in [1], Lemma II.1.2.).

Given such a submodule-closed subcategory \mathcal{C} , one may ask whether it is finite or not, and in case it is infinite, it should be of interest to look at the corresponding minimal subcategories.

Example 1 (Kerner–Takane). Let Λ be a connected hereditary artin algebra of infinite representation type. The preprojective component of $\text{mod } \Lambda$ is a minimal subcategory.

Proof. Kerner–Takane [2], Lemma 6.3 have shown: For every $b \in \mathbb{N}$, there is $n = n(b) \in \mathbb{N}$ with the following property: If P, P' are indecomposable projective modules, then $\tau^{-i} P'$ is cogenerated by $\tau^{-j} P$, for all $0 \leq i \leq b$ and $n \leq j$. Assume that \mathcal{C} is the additive subcategory given by an infinite set of indecomposable preprojective modules. We claim that the cogeneration closure of \mathcal{C} contains all the preprojective modules X . Indeed, let $X = \tau^{-b} P'$ with P' indecomposable projective. Choose a corresponding $n(b)$. Since \mathcal{C} contains infinitely many isomorphism classes of indecomposable preprojective modules, there is some $C = \tau^{-j} P$ in \mathcal{C} with $n \leq j$ and P indecomposable projective. According to Kerner–Takane, X is cogenerated by C .

Recall that an algebra Λ is said to be *tame concealed* provided it is the endomorphism ring of a preprojective tilting module of a tame hereditary algebra. \square

Example 2. Any tame concealed algebra Λ has a unique minimal subcategory \mathcal{C} , namely the subcategory of all preprojective modules.

Proof. Let k be a field and Λ a finite-dimensional k -algebra which is tame concealed. Let \mathcal{C} be an infinite submodule-closed subcategory of $\text{mod } \Lambda$. We want to show that \mathcal{C} contains infinitely many isomorphism classes of indecomposable preprojective modules.

According to Theorem 2(b) and (c), for any indecomposable module $C \in \mathcal{C}$, there exists an infinite inclusion sequence of indecomposable modules in \mathcal{C} which starts with C . This shows that C cannot be preinjective, since an indecomposable preinjective module for a tame concealed algebra has only finitely many successors. Thus, all the modules in \mathcal{C} are preprojective or regular.

Next, assume that \mathcal{C} contains infinitely many indecomposable regular modules. If they are of bounded length, then the proof of Brauer–Thrall 1 presented in Appendix A of [3] yields arbitrar-

ily large indecomposable modules M cogenerated by these regular modules, and the modules M constructed in this way have to be preprojective. It remains to consider the case that \mathcal{C} contains arbitrarily large indecomposable regular modules.

Recall that an indecomposable Λ -module H is said to be *homogeneous* provided its Auslander–Reiten translate τH is isomorphic to H . Note that if H is a homogeneous indecomposable module, then $\text{Hom}(P, H) \neq 0$ for all indecomposable preprojective modules P . We choose two indecomposable homogeneous Λ -modules H, H' which belong to different Auslander–Reiten components. Let b be an upper bound for the k dimension of all the vector spaces $\text{Ext}^1(Q, H)$ and $\text{Ext}^1(Q, H')$, where Q is a submodule of an indecomposable injective Λ -module (clearly, such a bound exists).

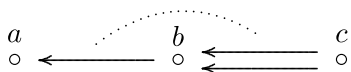
Now, let R be an indecomposable regular module in \mathcal{C} of length r , and let R' be its regular socle. Let $f' : R' \rightarrow Q'$ be a non-zero map with Q' indecomposable injective and let $f : R \rightarrow Q'$ be an extension of f' . Let Q be the image of f . By construction, R' is not contained in the kernel X of f , and therefore X has no non-zero regular submodule. It follows that X is a direct sum of say t indecomposable preprojective modules X_i . At least one of H, H' , say H , will belong to a different Auslander–Reiten component than R , and thus $\text{Hom}(R, H) = 0 = \text{Ext}^1(R, H)$. We apply $\text{Hom}(-, H)$ to the exact sequence $0 \rightarrow X \rightarrow R \rightarrow Q \rightarrow 0$, and obtain

$$\text{Hom}(R, H) \rightarrow \text{Hom}(X, H) \rightarrow \text{Ext}^1(Q, H) \rightarrow \text{Ext}^1(R, H)$$

with first and last term being zero, thus the k -spaces $\text{Hom}(X, H)$ and $\text{Ext}^1(Q, H)$ are isomorphic. In particular, we see that the k dimension of $\text{Hom}(X, H)$ is bounded by b . Since X is the direct sum of t indecomposable preprojective modules, and $\text{Hom}(P, H) \neq 0$ for any indecomposable preprojective module P , it follows that $t \leq b$. Let q be the maximal length of an indecomposable injective Λ -module, then $|X| \geq r - q$. Assume that all indecomposable direct summands X_i have length $|X_i| < \frac{1}{b}(r - q)$. Then $|X| = |\bigoplus_i X_i| < b \cdot \frac{1}{b}(r - q) = r - q$, a contradiction. This shows that at least one of the modules X_i has length $|X_i| \geq \frac{1}{b}(r - q)$. Since by assumption r is not bounded, also $\frac{1}{b}(r - q)$ is not bounded.

Thus, we have shown that \mathcal{C} contains infinitely many isomorphism classes of indecomposable preprojective modules, and therefore the intersection \mathcal{C}'' of \mathcal{C} with the preprojective component is infinite. The minimality of \mathcal{C} implies that \mathcal{C} contains only preprojective modules. On the other hand, as in Example 1, the subcategory of all preprojective modules can be shown to be minimal. \square

Remark. Preprojective components are always submodule-closed, but in general an infinite preprojective component \mathcal{P} does not have to be minimal. First of all, \mathcal{P} may contain indecomposable injective modules, whereas this cannot happen for a minimal subcategory, as part (b) of Theorem 2 shows. But also preprojective components without indecomposable injective modules may not be minimal. For example, consider the algebra with quiver



with one zero relation (thus, the indecomposable projective module P_a corresponding to the vertex a is simple, the radical of P_b is equal to P_a and the radical of P_c is the direct sum of P_b and the simple factor module of P_b). Then the preprojective component \mathcal{P} contains indecomposables

which are faithful, but also countable many indecomposables X with $X_a = 0$. Clearly, the subcategory of \mathcal{P}' of all modules P in \mathcal{P} with $P_a = 0$ is a proper subcategory which is both infinite and submodule-closed (and actually, \mathcal{P}' is minimal).

Example 3. Let \mathcal{I} be a two-sided ideal in Λ . The category of Λ -modules annihilated by \mathcal{I} is obviously submodule-closed and of course equivalent (or even equal) to the category of all Λ/\mathcal{I} -modules. If Λ/\mathcal{I} is representation-infinite, then $\text{mod } \Lambda/\mathcal{I}$ will contain a minimal subcategory. Consider for example the generalized Kronecker-algebra $K(3)$ with three arrows α, β, γ . The one-dimensional ideals of $K(3)$ correspond bijectively to the elements of the projective plane \mathbb{P}^2 , say $a = (a_0 : a_1 : a_2) \in \mathbb{P}^2$ yields the ideal $\mathcal{I}_a = \langle a_0\alpha + a_1\beta + a_2\gamma \rangle$. Let \mathcal{C}_a be the additive subcategory of $\text{mod } K(3)$ of all preprojective $K(3)/\mathcal{I}_a$ -modules. Then these are pairwise different subcategories (the intersection of any two of these subcategories is the subcategory of semisimple projective modules). In particular, *if the base field is infinite, there are infinitely many subcategories in $\text{mod } K(3)$ which are minimal.* (Note that the preprojective $K(3)$ -modules provide a further minimal subcategory.)

The minimal subcategories exhibited here can be distinguished by looking at the corresponding annihilators (the annihilator of a subcategory \mathcal{C} is the ideal of all the elements $\lambda \in \Lambda$ which annihilate all the modules in \mathcal{C}). The next example will show that usually there are also different minimal subcategories which have the same annihilator. Note that a submodule-closed subcategory \mathcal{C} has zero annihilator if and only if all the projective modules belong to \mathcal{C} .

Example 4. Here is an artin algebra Λ with different minimal categories containing all indecomposable projective modules. Consider the hereditary algebra Λ with quiver Q .

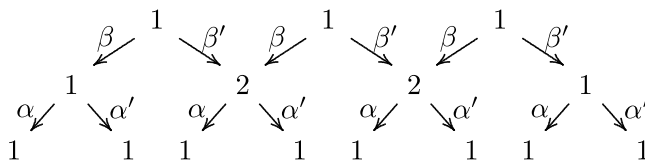
$$\begin{array}{ccccc} a & & \alpha & & b & & \beta & & c \\ \circ & \xleftarrow{\quad} & & \xleftarrow{\quad} & \circ & \xleftarrow{\quad} & & \xleftarrow{\quad} & \circ \\ & \alpha' & & \beta' & & & & & \end{array}$$

We denote by $Q(ab)$ the full subquiver of Q with vertices a, b , by $Q(bc)$ that with vertices b, c .

As we know, the preprojective component \mathcal{C} of $\text{mod } \Lambda$ is a minimal subcategory. Of course, it contains all the projective Λ -modules, but it contains also, for example, the indecomposable Λ -module X with dimension vector $(3, 2, 0)$; note that the restriction of X to $Q(ab)$ is indecomposable and neither projective nor semisimple.

Second, let \mathcal{D} be the full subcategory of $\text{mod } \Lambda$ consisting of all the Λ -modules such that the restriction to $Q(ab)$ is projective and the restriction to $Q(bc)$ is preprojective. Clearly, \mathcal{D} is submodule-closed, and it is obviously infinite: If Y is a Λ -module with $Y_a = 0$, define \underline{Y} as follows: the restrictions of Y and \underline{Y} to $Q(bc)$ should coincide, whereas the restriction of \underline{Y} to $Q(ab)$ should be a direct sum of indecomposable projectives of length 3; in particular, $\underline{Y}_a = Y_b^2$. By $Y \mapsto \underline{Y}$ we obtain an embedding of the category of preprojective Kronecker modules into \mathcal{D} , which yields all the indecomposable modules in \mathcal{D} but the simple projective one. It follows easily that \mathcal{D} is minimal. Of course, $\mathcal{D} \neq \mathcal{C}$, and note that also \mathcal{D} contains all the projective Λ -modules.

We can exhibit even a third minimal subcategory which contains all the projective Λ -modules, by looking at the full subcategory \mathcal{E} of Λ -modules such that the restriction to $Q(ab)$ is the direct sum of a projective and a semisimple module, whereas the restriction to $Q(bc)$ is projective. Again, clearly \mathcal{E} is submodule-closed. In order to construct an infinite family of indecomposable modules in \mathcal{E} , we use covering theory: The following quiver is part of the universal cover \hat{Q} of Q



and the numbers inserted form the dimension vector for a two-parameter family of indecomposable modules M . If we require in addition that the maps α and α' starting at the same vertex have equal kernels, then there is a unique isomorphism class $M = Y_3$ with this dimension vector. In a similar way, we can construct for any natural number n an indecomposable representation Y_n of \widehat{Q} of length $2 + 5n$ (with top of length n). The kernel condition assures that the Λ -module which is covered by $M = Y_3$, or more generally, by Y_n , belongs to \mathcal{E} (note that the kernel condition means that the restriction of M to any subquiver of type $\widetilde{\mathbb{D}}_4$ has socle of length 3). If \mathcal{E}' is a minimal subcategory inside \mathcal{E} , then \mathcal{E}' is different from \mathcal{C} and \mathcal{D} .

Remark. The Λ -module covered by Y_1 is indecomposable projective and has Gabriel–Roiter measure $(1, 3, 7)$, this is the measure I_3 for Λ . One may show that the Λ -module covered by Y_2 has Gabriel–Roiter measure $(1, 3, 7, 12)$ and that this is the measure I_4 . For $t \geq 5$, the measures I_t are not yet known; it would be interesting to decide whether the intersection of the take-off part of $\text{mod } \Lambda$ and \mathcal{E} is infinite or not.

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